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Hamilton Spaces of Order $k \geq 1$

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A suitable "dual" for the k-acceleration bundle ($T^k M$, π^k, M) is the fibered bundle ($T^{k-1}M \times_M T^*M$). The mentioned bundle carries a canonical presymplectic structure and k canonical Poisson structures. By means of this "dual" we define the notion of Hamilton spaces of order k, whose total space consists of points x of the configuration space M, accelerations of order 1, . . . , $k - 1$, $y^{(1)}$, . . . , $y^{(k-1)}$, and momenta p. Some remarkable Hamiltonian systems are pointed out. There exists a Legendre mapping from the Lagrange spaces of order k to the Hamilton space of order k.

INTRODUCTION

The notion of a Hamilton space was introduced by the author in refs. 4 and 5. It refers to a pair $H^n = (M, H(x, p))$, where M is a smooth n-dimensional manifold and H is a regular Hamiltonian, that is, a smooth function on the cotangent manifold T^*M, whose Hessian with respect to the momenta p_i is nonsingular. The space H^n has a canonical symplectic structure and, accordingly, a canonical Poisson structure. The regularity of the Hamiltonian H allows us to view the space H^n as the dual, via a Legendre mapping, of a Lagrange space $L^n = (M, L(x, y))$ [1, 5, 8, 10].

The notion of Lagrange space of higher order $k \geq 1$, $L^{(k)n} = (M, L)$, was defined some years ago [7], L being a regular Lagrangian of order k. But up to now no definition for the notion of higher order Hamilton space has been proposed. The reason is that it is not simple to find a *dual* of a Lagrange space of order k, $L^{(k)n}$. Here, duality is not algebraic, but refers to the existence of a local diffeomorphism (a Legendre mapping) between two spaces.

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In the present paper we propose a suitable "dual" of the k-acceleration bundle T^kM. This is the fibered bundle $T^{k-1}TM \times_M T^*M$ over M. We show that the total space of the latter has a canonical presymplectic structure as well as k canonical Poisson structures. Thus the notion of a higher order Hamiltonian space appears in a natural way. It is a pair $H^{(k)n} = (M, H)$, where H: $T^{k-1}LM \times_M T^*M \to R$ is a regular Hamiltonian depending on the point $x \in M$, the accelerations of order $1, 2, \ldots, k-1, y^{(1)}, \ldots, y^{(k-1)}$, and the momenta $p \in T^*M$. The spaces $H^{(k)n}$ have the following important properties:

(a) $\dim H^{(k)n} = \dim L^{(k)n}$.
(b) $H^{(k)n}$ has a canonical presymplectic structure and a number k of canonical Poisson structures.
(c) The spaces $H^{(k)n}$ and $L^{(k)n}$ are local diffeomorphic, via a Legendre transformation.

Of course, these properties hold in the case $k = 1$. The geometry of the Hamilton spaces of order $k \geq 1$ is a natural extension of the geometry of Hamilton spaces $H^n = (M, H(x, p))$.

1. THE DUAL OF THE *k*-OSCULATING BUNDLE

Let M be a real, C^∞-differentiable manifold M of dimension n and (TM, π, M), (T^*M, π^*, M) its tangent and cotangent bundle, respectively. We consider the bundle of accelerations of order k, (T^kM, π^k, M), which is identified with the osculating bundle of order k, (Osc^kM, π^k, M) [8]. The points $u \in Osc^kM$ are of the form $u = (x, y^{(1)}, \ldots, y^{(k)})$ with the canonical coordinates $(x^i, y^{(1)i}, \ldots, y^{(k)i})$. Latin indices $i, j, k, \ldots$ run over the set $\{1, 2, \ldots, n\}$, and the summation convention will be used.

For the bundles $(Osc^{k-1}M, \pi^{k-1}, M)$ and (T^*M, π^*, M) the fibered product

$$(Osc^{k-1}M \times_M T^*M, \pi^{*k}, M) \tag{1.1}$$

can be considered. The projection π^{*k}: $Osc^{*k}M \to M$, where $Osc^{*k}IM = Osc^{k-1}XM \times_M T^*M$, is given by $\pi^{*k}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = x$. The points (x, p) belong to the manifold T^*M and their local coordinate are (x^i, p_i).

Thus, a point $u = (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in Osc^{*k}M$ consists of a point x, accelerations $y^{(1)}, \ldots, y^{(k-1)}$ of order $1, \ldots, k-1$, and a momentum p (using terms from in analytical mechanics).

The manifold $Osc^{*k}M$ will be called the "dual space" of the total space of the k-osculating bundle Osc^kM. We say that Osc^kM and $Osc^{*k}M$ are dual to each other since between them there exists a local Legendre diffeomorphism. In this sense and not in an algebraic one, we say that the bundle $(Osc^{*k}M,$

π^{*k}, M) is dual to the bundle (Osc^k, π^k, M). For $k = 1$, ($Osc^{*1}M$, π^{*1}, M) is identified with the cotangent bundle (T^*M, π^*, M).

The following diagram, where the arrows indicate natural projections, is commutative:

$$\begin{array}{ccccc} & & Osc^{*k}M & & \\ & \swarrow & & \searrow & \\ Osc^{k-1}M & & \downarrow & & T^*M \\ & \searrow & & \swarrow & \\ & & M & & \end{array}$$

A change of local coordintes on the manifold $Osc^{*k}M$ is given by

$$\begin{aligned} \tilde{x}^i &= \tilde{x}^i(x^1, \ldots, x^n), \qquad \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0 \\ \tilde{y}^{(1)i} &= \frac{\partial \tilde{x}^i}{\partial x^j} y^j \\ \cdots (k-1)\tilde{y}^{(k-1)i} &= \frac{\partial \tilde{y}^{(k-2)i}}{\partial x^j} y^{(1)j} + \cdots + (k-1) \frac{\partial \tilde{y}^{(k-2)i}}{\partial y^{(k-2)j}} y^{(k-1)j} \qquad (1.2) \\ \tilde{p}_i &= \frac{\partial x^j}{\partial \tilde{x}^i} p_j \end{aligned}$$

and the following identities hold [7]:

$$\frac{\partial \tilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(\alpha+1)}}{\partial y^{(1)j}} = \cdots = \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1-\alpha)j}}; \qquad \alpha = 0, \ldots, k-2; \quad y^{(0)} = x \tag{1.2'}$$

Using formulas (1.2), we can introduce the following differential forms on the manifold $Osc^{*k}M$:

$$\begin{aligned} \omega &= p_i\, dx^i \qquad (1.3) \\ \theta &= d\omega = dp_i \wedge dx^i \end{aligned}$$

From (1.2) it follows that $\tilde{p}_i\, d\tilde{x}_i = p_i\, dx^i$. Thus the following assertions are clear.

Theorem 1.1.
1. The forms ω and θ are globally defined on $Osc^{*k}M$.
2. $d\theta = 0$, $\operatorname{rank}\|\theta\| = 2n$.
3. θ is a canonical presymplectic structure on the manifold $Osc^{*k}M$.

Let us consider the systems of Poisson brackets: for any $f, g \in \mathcal{F}$ $(Osc^{*k}M)$,

$$\{f, g\}_\alpha = \frac{\partial f}{\partial y^{(\alpha)i}}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial y^{(\alpha)i}}\frac{\partial f}{\partial p_i}, \qquad \alpha = 0, 1, \dots, k-1; \quad y^{(0)} = x \tag{1.4}$$

Theorem 1.2. Every bracket $\{\cdot\}_\alpha$ ($\alpha = 0, \dots, k-1$) defines a canonical Poisson structure on the manifold $Osc^{*k}M$.

Proof. First, it is not difficult to see that for $\alpha = 0, 1, \dots, k-1$, $\{f, g\}_\alpha \in \mathcal{F}(Osc^{*k}M)$ and $\{\tilde{f}, \tilde{g}\}_\alpha = \{f, g\}_\alpha$.

Indeed, by means of (1.2) we have

$$\frac{\partial f}{\partial y^{(\alpha)i}} = \frac{\partial \tilde{y}^{(\alpha)m}}{\partial y^{(\alpha)i}}\frac{\partial \tilde{f}}{\partial \tilde{y}^{(\alpha)m}} + \frac{\partial \tilde{y}^{(\alpha+1)m}}{\partial y^{(\alpha)i}}\frac{\partial \tilde{f}}{\partial \tilde{y}^{(\alpha+1)m}} + \cdots + \frac{\partial \tilde{y}^{(k-1)m}}{\partial y^{(\alpha)i}}\frac{\partial \tilde{f}}{\partial \tilde{y}^{(k-1)m}} + \frac{\partial \tilde{p}_m}{\partial y^{(\alpha)i}}\frac{\partial \tilde{f}}{\partial \tilde{p}_m}$$

$$\frac{\partial g}{\partial p_i} = \frac{\partial x^i}{\partial \tilde{x}^s}\frac{\partial \tilde{g}}{\partial \tilde{p}_s}$$

Using (1.2′), we can write first of the previous formulas as

$$\frac{\partial f}{\partial y^{(\alpha)i}} = \frac{\partial \tilde{x}^m}{\partial x^i}\frac{\partial \tilde{f}}{\partial \tilde{y}^{(\alpha)m}} + \frac{\partial \tilde{y}^{(1)m}}{\partial x^i}\frac{\partial \tilde{f}}{\partial \tilde{y}^{(\alpha+1)m}} + \cdots + \frac{\partial \tilde{y}^{(k-1-\alpha)m}}{\partial x^i}\frac{\partial \tilde{f}}{\partial \tilde{y}^{(k-1)m}} + \frac{\partial \tilde{p}_m}{\partial y^{(\alpha)i}}\frac{\partial \tilde{f}}{\partial \tilde{p}_m}$$

Now, taking into account the identities

$$\frac{\partial \tilde{p}_m}{\partial y^{(\alpha)i}} = 0 \quad \text{for } \alpha \neq 0, \qquad \frac{\partial \tilde{p}_m}{\partial x^i}\frac{\partial x^i}{\partial \tilde{x}^s} = 0$$

$$\frac{\partial \tilde{y}^{(\beta)m}}{\partial x^i}\frac{\partial x^i}{\partial \tilde{x}_s} = \frac{\partial \tilde{y}^{(\beta)m}}{\partial \tilde{x}^s} = 0 \qquad \text{for} \quad \beta = 1, 2, \dots, k-1$$

we obtain

$$\frac{\partial f}{\partial y^{(\alpha)i}}\frac{\partial g}{\partial p_i} = \frac{\partial \tilde{f}}{\partial \tilde{y}^{(\alpha)i}}\frac{\partial \tilde{g}}{\partial \tilde{p}_i}, \qquad \text{for } \alpha = 0, \dots, k-1; \quad y(0) = x$$

Consequently,

$$\{f, g\}_\alpha = \{\tilde{f}, \tilde{g}\}_\alpha$$

Now it is not difficult to prove that the brackets $\{f, g\}_\alpha$ (i) are R-linear in every argument, (ii) are skew-symmetric: $\{f, g\}_\alpha = -\{g, f\}_\alpha$, and (iii) satisfy the Jacobi identities

$$\{\{f, g\}_\alpha, h\}_\alpha + \{\{g, h\}_\alpha, f\}_\alpha + \{\{h, f\}_\alpha, g\}_\alpha = 0, \quad \alpha = 0, 1, \ldots, k-1$$

and the mapping $\{f, \cdot\}_\alpha$: $\mathcal{F}(Osc^{*k}M) \to \mathcal{F}(Osc^{*k}M)$is a derivation of the functions algebra $\mathcal{F}(Osc^{*k}M)$.

The previous theorems allow us to study Hamiltonian systems over the manifold $Osc^{*k}M$.

2. THE HAMILTONIAN SYSTEMS OF ORDER k

As usual [7], we set $\tilde{O}sc^{*k}M = Osc^{*k}M \setminus \{0\}$, where 0 means the zero section of the projection π^{*k}.

Definition 2.1. A mapping H: $Osc^{*k}M \to R$ is called a *differentiable Hamiltonian of order k* if H is a differentiable function on $\tilde{O}sc^{*k}M$, and it is continuous on the zero section.

Thus, if $H(u) = H(x, y^{(1)}, \ldots, y^{(k-1)}, p)$ is a function of the particle x, the accelerations of order 1, 2, . . . , $k - 1$, and the momenta p_i, it will be a differentiable Hamiltonian of order k if this function is differentiable on the manifold $\tilde{O}sc^{*k}M$; it is continuous at the points $(x, 0, \ldots, 0, 0)$.

Definition 2.2. A Hamiltonian system of order k is a triple $(Osc^{*k}M, \theta, H)$, where θ is a presymplectic structure on $Osc^{*k}M$ and H is a differentiable Hamiltonian on the manifold $Osc^{*k}M$.

In the particular case $k = 1$, where θ is the canonical symplectic structure over T^*M, we have the classical Hamiltonian systems.

If θ is the presymplectic structure on $Osc^{*k}M$ given by (1.3.) and H is a differentiable Hamiltonian on $Osc^{*k}M$, we obtain an important Hamiltonian system, which can be studied by a method of Gotay [2]. In this case, the Poisson structure $\{\cdot\}_0$ will be considered. We follow here another way [11,12], introducing k-induced Hamiltonian systems $\{E_\alpha, \theta_\alpha, H_\alpha\}$ $(\alpha = 0, 1, \ldots, k - 1)$ as follows.

Let us consider the section Σ_0 of the projection $Osc^{*k}M \to T^*M$ (from the above diagram), defined by

$$\Sigma_0 = \{(x, y^{(1)}, \ldots, y^{(k-1)}, p) \in Osc^{*k}M \big| y^{(1)} = \cdots = y^{(k-1)} = 0\}.$$

It is an immersed submanifold of the manifold $Osc^{*k}M$. We denote by H_0 the restriction to Σ_0 of a differentiable Hamiltonian H on $Osc^{*k}M$ and by θ_0 the restriction of the 2-form θ of (1.3). Of course, $\dim \Sigma_0 = 2n$, θ_0 is a canonical symplectic structure, and H_0 is a differentiable Hamiltonian on Σ_0. Consequently, the triple $(\Sigma_0, \theta_0, H_0)$ is a Hamiltonian system.

Now, we can prove the following:

Theorem 2.1:

1. The triple $(\Sigma_0, \theta_0, H_0)$ is a Hamiltonian system, θ_0 being a symplectic structure on Σ_0.

2. There exists a unique vector field X_{H_0} determined by the equation

$$i_{X_{H_0}}\theta_0 = -dH_0 \tag{2.1}$$

3. The integral curves of the vector field X_{H_0} are given by the canonical equations

$$\frac{dx^i}{dt} = \frac{\partial H_0}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H_0}{\partial x^i} \qquad (\text{on } \Sigma_0) \tag{2.2}$$

4. The following equation holds:

$$\{f, g\}_0 = \theta_0(X_f, X_g) \qquad \forall f, g \in \mathcal{F}(\Sigma_0) \tag{2.3}$$

where $\{f, g\}_0$ is the Poisson structure, (1.4).

The previous result is also valid for the Poisson structure $\{\cdot\}_\alpha$ $(\alpha = 1, \ldots, k - 1)$.

Indeed, let Σ_α be the immersed submanifold of the fiber $(\pi^{*k})^{-1}(x_0) \subset Osc^{*k}M$ in a fixed point $x_0 \in M$. Than Σ_α is defined by

$$\begin{aligned} \Sigma_\alpha = \{&(x, y^{(1)}, y^{(2)}, \ldots, y^{(k-1)}, p) \\ &\in Osc^{*k}M \big| (x = x_0, y^{(\beta)} = 0, \beta \neq \alpha, y^{(\alpha)} = y^{(\alpha)}, p = p) \\ &(\alpha = 1, \ldots, k-1)\} \end{aligned} \tag{2.4}$$

Consequently, Theorem 2.1 can be proved for the Hamiltonian systems $(\Sigma_\alpha, \theta_\alpha, H_\alpha)$ where H_α is the restriction of the differentiable Hamiltonian $H(x, y^{(1)}, \ldots, y^{(k-1)}, p)$ to the submanifold Σ_α and

$$\theta_\alpha = dp_i \wedge dy^{(\alpha)i}, \qquad \alpha = 1, \ldots, k-1 \tag{2.5}$$

Obviously, denoting

$$\omega_\alpha = p_i\, dy^{(\alpha)i} \qquad \alpha = 1, \ldots, k-1$$

we have $k - 1$ 1-forms, every one being defined on the submanifold Σ_α. It follows that $\theta_\alpha = d\omega_\alpha$ $(\alpha = 1, \ldots, k-1)$.

As in the previous theorem, we have the following result:

Theorem 2.2:

1. For each $(\alpha = 1, \ldots, k-1)$, the triple $(\Sigma_\alpha, \theta_\alpha, H_\alpha)$ is a Hamiltonian system, θ_α given by (1.4) being a symplectic structure on the submanifold Σ_α.

2. There exists a unique vector field X_{H_α} determined by the equation

$$i_{X_{H_\alpha}} \theta_\alpha = -\, dH_\alpha \tag{2.6}$$

3. The integral curves of the vector field X_{H_α} are the "canonical equations"

$$\frac{dy^{(\alpha)i}}{dt} = \frac{\partial H_\alpha}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H_\alpha}{\partial y^{(\alpha)i}} \qquad (\text{on } \Sigma_\alpha) \tag{2.7}$$

4. The following equation holds:

$$\{f, g\}_\alpha = \theta_\alpha(X_f, X_g) \qquad \forall f, g \in \mathcal{F}(\Sigma_\alpha) \tag{2.8}$$

$\{f, g\}_\alpha$ ($\alpha = 1, \ldots, k - 1$) being given by the Poisson structures (1.4).

Of course, this theory is valid in the case when the differentiable Hamiltonian of order k, H, is regular.

3. THE NOTION OF HAMILTON SPACE OF ORDER *k*

The Hessian matrix of a differentiable Hamiltonian of order k, $H(x, y^{(1)}, \ldots, y^{(k-1)}, p)$ with respect to the momenta p_i, has the enteries

$$g^{ij} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j} \qquad \text{on} \quad \tilde{O}sc^{*k}M \tag{3.1}$$

We can prove without difficulty that g^{ij} is a distinguished contravariant symmetric tensor field.

We say that H is regular if

$$\text{rank}\, \|g^{ij}\| = n = \dim M, \qquad \text{on} \quad \tilde{O}sc^{*k}M \tag{3.1$'$}$$

Definition 3.1. A Hamilton space of order k ($k \in \mathbb{N}^*$) is a pair $H^{(k)n} = (M, H(x, y^{(1)}, \ldots, y^{(k-1)}, p))$ formed by a smooth, real, n-dimensional manifold M and a differentiable regular Hamiltonian H of order k, for which the d-tensor field g^{ij} has a constant signature on $Osc^{*k}M$.

It is easy to prove the following:

Theorem 3.1. If the base manifold M is paracompact, then on the manifold $Osc^{*k}M$ there exist Hamiltonians H such that the pairs (M, H) are Hamilton spaces of order k.

Example. Let $F^{(k-1)n} = (M, F\,(x, y^{(1)}, \ldots, y^{(k-1)}))$ be a Finsler space of order $k - 1$ [8]. F exists if M is a paracompact manifold. If $a_{ij}(x, y^{(1)}, \ldots, y^{(k-1)})$ is the fundamental tensor field of the space $F^{(k-1)n}$, then

$$H(x, y^{(1)}, \ldots, y^{(k-1)}, p) = a^{ij}\,(x, y^{(1)} \ldots, y^{(k-1)}) p_i p_j \tag{3.2}$$

where a^{ij} is the contravariant tensor associated with a_{ij}, is the fundamental function of a Hamilton space of order k.

For a Hamilton space $H^{(k)n} = (M, H)$, H is called the fundamental function and g^{ij} its fundamental tensor field.

The geometry of the Hamilton spaces of order k can be studied as a natural extension of the geometry of Hamilton space of order 1 [1, 4, 5, 10].

Some special classes of spaces $H^{(k)n}$ are as follows.

1. The Riemannian spaces $H^{(k)n} = (M, H)$ defined as spaces $H^{(k)n}$ for which the fundamental function $H(x, y^{(1)}, \ldots, y^{(k-1)}, p)$ is 2-homogeneous with respect to p_i and its fundamental tensor field g^{ij} does not depend on the variables p_i. We denote them by $\mathcal{R}^{(k)n}$.
2. The Cartan spaces of order k are the Hamilton spaces of order k, for which the fundamental function H is 2-homogeneous with respect to the momenta p_i. We denote them by $\mathcal{C}^{(k)n} = (M, H(x, y^{(1)}, \ldots, y^{(k-1)}, p))$.

A more general class of spaces is given by the generalized Hamilton spaces of order k.

They are defined as pairs $GH^{(k)n} = (M, g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p_i))$, where g^{ij} is a distinguished tensor fields [9] which has the properties that $\operatorname{rank}\|g^{ij}\| = n$ on $\tilde{O}sc^{*k}M$ and g^{ij} has a constant signature.

Example. If $\gamma^{ij}(x)$ is the contravariant tensor of the metric tensor $\gamma_{ij}(x)$ of a Riemannian space on the manifold M and $n(x, y^{(1)}, \ldots, y^{(k-1)}, p) > 1$ is a function (refractive index) on $Osc^{*k} M$, then the spaces $GH^{(k)n} = (M, g^{ij})$, with

$$g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = \gamma^{ij}(x) + \left(1 - \frac{1}{n^2(x, y^{(1)}, \ldots, y^{(k}{}_{-1)}, p)}\right) p^i p^j \tag{3.3}$$

where $p^i = \gamma^{ij}(x)p_j$, are generalized Hamilton spaces of order k.

For the space $GH^{(k)n}$ there is no fundamental function H such that $g^{ij} = \frac{1}{2}\, \partial^2 H/\partial p_i \partial p_j$.

The spaces $GH^{(k)n}$ were suggested by Synge's metric of relativistic optics [9].

Evidently, the previous classes of Hamilton spaces satisfy the inclusions

$$\{\mathcal{R}^{(k)n}\} \subset \{\mathcal{C}^{(k)n}\} \subset \{\mathrm{H}^{(k)n}\} \subset \{GH^{(k)n}\}$$

Similar inclusions hold for the Lagrange spaces of order k [7]:

$$\{\mathcal{R}^{(k)n}\} \subset \{F^{(k)n}\} \subset \{L^{(k)n}\} \subset \{GL^{(k)n}\}$$

where $F^{(k)n}$ are Finsler spaces of order k [8].

The relation between the Lagrange spaces of order k,

$$L^{(k)n} = (M, L(x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}))$$

and the Hamilton spaces of order k, $H^{(k)n} = (M, H(x, y^{(1)}, \ldots, y^{(k-1)}, p)$, is given by the Legendre mapping Leg: $L^{(k)n} \to H^{(k)n}$ defined by

$$\begin{aligned} \text{Leg:}\quad & (x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}) \\ & \in Osc^k M \to (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in Osc^{*k} M \end{aligned} \tag{3.4}$$

where

$$p_i = \frac{1}{2} \frac{\partial L}{\partial y^{(k)i}} \tag{3.4$'$}$$

We have the following result:

Theorem 3.2. The Legendre mapping Leg given by (3.4) and (3.4′) is a local diffeomorphism between the total spaces of Lagrange space $L^{(k)n}$ and the Hamilton space $H^{(k)n}$.

Indeed, the determinant of the Jacobian matrix of the mapping Leg coincides with the determinant of the matrix $\|a_{ij}\|$, where $a_{ij}(x, y^{(1)}, \ldots, y^{(k)})$ is the fundamental tensor of the space $L^{(k)n}$. QED

Consequently, one can study the geometry of the Hamilton spaces of order k directly as an extension of the case $k = 1$ and by using the Legendre transformations (3.4), (3.4′) since the geometry of the spaces $L^{(k)n}$ is known [7].

4. CONCLUSIONS

The Lagrange spaces of order $k \geq 1$, $L^{(k)n} = (M, L\ (x, y^{(1)}, \ldots, y^{(k)}))$, are defined by the regular Lagrangians L which depend on points x of the configuration space M and accelerations $y^{(1)}, \ldots, y^{(k)}$ of order $1, \ldots, k$, respectively [7]. The space $L^{(k)n}$ is a natural extension of the notion of Lagrange space of order 1, $L^n = (M, L\ (x, y))$.

The problem is to construct the dual notion of Hamilton space of order k, $H^{(k)n} = (M, H)$. It must have the following properties:

1. $\dim L^{(k)n} = \dim H^{(k)n}$.
2. On $H^{(k)n}$ there exist a canonical presymplectic structure and a canonical Poisson structure.
3. There exists a local diffeomorphism between the total spaces of $L^{(k)n}$ and $H^{(k)n}$.

The "dual space" of $L^{(k)n}$ obtained by means of Jacobi–Ostrogradski momenta [3, 7] does not satisfy the previous strong conditions.

In the present paper this problem is solved by means of the fibered bundle (1.1), where $(Osc^{k-1} M, \pi^{k-1}, M)$ is the $(k-1)$-osculator bundle (or $k-1$ acceleration bundle) and (T^*M, π^*, M) is the cotangent bundle of the manifold M.

The main results concerning the dual of a k-osculator bundle, Hamiltonian systems of order k, and Hamilton spaces of order k, as well as the Legendre transformation, are given by Theorems 1.1, 1.2, 2.1, 2.2, and 3.2.

The geometry of the Hamilton spaces of order k is a natural extension of the known geometry of Hamilton space $H^n = (M, H(x, p))$ taking into account the Legendre mapping (3.4), (3.4′). Applications in analytical mechanics of the higher order Lagrangians can be done starting from Examples 3.2 and 3.3.

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